

Testing Composite Null Hypothesis Based on S -Divergences

Abhik Ghosh¹, Ayanendranath Basu²

Indian Statistical Institute

Abstract

We present a robust test for composite null hypothesis based on the general S -divergence family. This requires a non-trivial extension of the results of Ghosh et al. (2015). We derive the asymptotic and theoretical robustness properties of the resulting test along with the properties of the minimum S -divergence estimators under parameter restrictions imposed by the null hypothesis. An illustration in the context of the normal model is also presented.

Keywords: Parameter Restriction, Composite Hypothesis Testing, Robustness, S -Divergence.

1. Introduction

Statistical tests for composite hypotheses are encountered all the time in all disciplines of applied sciences. For such composite hypotheses, the null parameter space is generally defined through some pre-specified restrictions and one needs to estimate the parameter value under those restrictions to perform the test. The most common and widely used statistical tool to solve this inferential problem is the classical likelihood ratio test (Neyman and Pearson, 1928, Wilks, 1938) which utilizes the maximum likelihood estimator of the parameters under given restrictions. However, the non-robust nature of such likelihood based solutions under misspecification of models and/or presence of outliers is well-known. So, there have been many attempts for developing robust alternative to the likelihood ratio test (LRT) with good asymptotic and robustness properties.

Ghosh et al. (2015) proposed a general family of tests of hypothesis for the simple null problem; these tests are based on the family of S -divergences (Ghosh et al., 2013a), and extend the idea of Basu et al. (2013a) who considered testing of hypothesis based on the density power divergence (Basu et al., 1998). In

Email addresses: abhianik@gmail.com (Abhik Ghosh), ayanbasu@isical.ac.in (Ayanendranath Basu)

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²Corresponding Author

the present paper we provide a non-trivial generalization of the Ghosh et al. (2015) paper, and the theoretical robustness properties described in this work also provide the theoretical underpinnings of the Basu et al. (2013b) tests as a special case. We also study the corresponding minimum divergence estimators under the restrictions imposed by the null.

In this paper, we presents the restricted minimum S -divergence estimator and its asymptotic distribution for both the discrete and continuous models. The main focus of the paper is on the theoretical robustness properties of the S -divergence based tests of composite hypothesis. For brevity in presentation, the proofs of all the results are provided in the online supplement to this paper.

2. The Restricted Minimum S -Divergence Estimators (RMSDE)

The S -divergence family has been recently introduced by Ghosh et al. (2013a) and contains several popular density-based divergences like the power divergence (PD) family of Cressie and Read (1984) and the density power divergence (DPD) family of Basu et al. (1998). For two densities g and f , it is defined in terms of two parameters $\gamma \in [0, 1]$ and $\lambda \in \mathbb{R}$ as

$$S_{(\gamma, \lambda)}(g, f) = \frac{1}{A} \int f^{1+\gamma} - \frac{1+\gamma}{AB} \int f^B g^A + \frac{1}{B} \int g^{1+\gamma}, \quad A \neq 0, B \neq 0, \quad (1)$$

where $A = 1 + \lambda(1 - \gamma)$ and $B = \gamma - \lambda(1 - \gamma)$. Whenever $A = 0$ or $B = 0$, the corresponding S -divergence measure is defined by the continuous limits of (1) as $A \rightarrow 0$ or $B \rightarrow 0$ respectively. Several properties and applications of the (unrestricted) minimum S -divergence estimators have been studied by Ghosh et al. (2013a,b), Ghosh (2014b), Ghosh and Basu (2014) and Ghosh et al. (2015). Here, we consider the minimum S -divergence estimators under some pre-specified parameter restrictions and study their asymptotic properties. The general theory of robustness for general minimum divergence estimators under parameter restrictions has been recently developed by Ghosh (2014a), which also contains the case of the S -divergence measures.

2.1. Definition and Estimating Equation

Consider the standard set-up of parametric inference, where we have a sample X_1, \dots, X_n from the true density g which is modeled by a parametric family of densities $\mathcal{F} = \{f_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$. We assume a set of r restrictions on the parameter θ given by

$$h(\theta) = 0, \quad (2)$$

such that the $p \times r$ matrix defined by $H(\theta) = \frac{\partial h(\theta)}{\partial \theta}$ exists with rank r and is a continuous function of θ .

The restricted minimum S -divergence estimator (RMSDE) of θ is to be obtained by minimizing $S_{(\alpha, \lambda)}(\hat{g}, f_\theta)$ subject to the constraints (2); here \hat{g} is some non-parametric estimator of the true density g ; this is given by the relative frequency of the values in the sample space for discrete models and by some kernel

density estimator for continuous models. See Ghosh (2014b) and Ghosh and Basu (2014) for corresponding descriptions in the unrestricted case. Then, using the method of Lagrange multipliers, the estimating equation of the RMSDE is given by

$$\left. \begin{aligned} \int K(\delta(x)) f_\theta^{1+\alpha}(x) u_\theta(x) dx + H(\theta) \lambda_n &= 0 \\ \text{and } h(\theta) &= 0 \end{aligned} \right\}, \quad (3)$$

where $\delta(x) = \delta_n(x) = \frac{\hat{g}(x)}{f_\theta(x)} - 1$, $K(\delta) = \frac{[(\delta+1)^A - 1]}{A}$ with $A = 1 + \lambda(1 - \alpha)$, λ_n is the vector of Lagrange multipliers, and $u_\theta = \frac{\partial}{\partial \theta} \log f_\theta$.

2.2. Asymptotic Distribution under Discrete Models

First we consider the case of discrete distributions, where both g and f_θ are densities with respect to some counting measure over the support $\chi = \{0, 1, 2, \dots\}$. We denote the relative frequency at any point x by $r_n(x)$. Then the RMSDE can be obtained as the solution of the estimating equation (3) with $\hat{g}(x)$ replaced by $r_n(x)$ and the integral replaced by the countable sum over the support χ .

To prove the asymptotic properties of the RMSDE under this set-up, we define $\tilde{\theta}^g = \arg \min_{\theta: h(\theta)=0} S_{(\alpha, \lambda)}(g, f_\theta)$, the restricted “best fitting parameter” under g . Then, we have the following result under the conditions (SA1)–(SA7) of Ghosh (2014b) provided in the Supplementary material.

Theorem 2.1. *Consider the above set-up of discrete models and assume that the conditions (SA1)–(SA7) of Ghosh (2014b) hold with respect to $\Theta_0 = \{\theta : h(\theta) = 0\}$ (instead of Θ). Then, we have the following:*

- (i) *There exists a consistent sequence $\tilde{\theta}_n$ of roots to the restricted minimum S -divergence estimating equations (3).*
- (ii) *Asymptotically, $\sqrt{n} (\tilde{\theta}_n - \tilde{\theta}^g) \sim N_p (0, \widetilde{P}_g \widetilde{V}_g \widetilde{P}_g)$, where the matrices \widetilde{P}_g and \widetilde{V}_g are as defined in Definition 1.1 in the Supplementary material.*

Next consider the particular case of the model density with $g = f_{\theta_0}$ for some $\theta_0 \in \Theta$ satisfying the given restriction (2). In this case, we have $\tilde{\theta}^g = \theta_0$ and hence $\sqrt{n} (\tilde{\theta}_n - \tilde{\theta}^g) \xrightarrow{D} N (0, \widetilde{P}_\alpha(\theta_0) \widetilde{V}_\alpha(\theta_0) \widetilde{P}_\alpha(\theta_0))$ asymptotically, where $\widetilde{P}_\alpha(\theta_0)$ and $\widetilde{V}_\alpha(\theta_0)$ are defined in Definition 1.1 in the Supplementary material. Interestingly the asymptotic distribution of the RMSDE at the model is also independent of the parameter λ defining the S -divergence measure — just as in the case of the unrestricted MSDE. It also coincides with the asymptotic distribution of the restricted minimum DPD estimators as obtained in Basu et al. (2013b) independently.

2.3. The Basu–Lindsay Approach for the RMSDE under Continuous Models

Now we consider the case of continuous models, where the densities g and f_θ are both continuous with respect to some common dominating measure. However, there is a clear incompatibility of measures between the data that are discrete and the assumed continuous model; hence we need to use kernel density estimator in place of \hat{g} in the estimating equation (3) which brings in several complications like bandwidth selection, curse of dimensionality etc. and complicated conditions are needed for the asymptotic results. The approach of Basu–Lindsay (Basu and Lindsay, 1994) helps us to avoid such complications by using convolution of the assumed model also by the same kernel; see Ghosh and Basu (2014) for several advantages of this approach and corresponding derivations in the unrestricted case.

Let us define the kernel density estimator g_n^* and the corresponding smoothed versions g^* and f_θ^* of the densities g and f_θ respectively:

$$\begin{aligned} g_n^*(x) &= \int W(x, y, h_n) dG_n(y) = \frac{1}{n} \sum_{i=1}^n W(x, X_i, h_n), \\ g^*(x) &= \int W(x, y, h) dG(y), \quad \text{and} \quad f_\theta^*(x) = \int W(x, y, h) dF_\theta(y), \end{aligned}$$

where $W(x, y, h_n)$ is a smooth kernel function with bandwidth h_n , G_n is the empirical distribution function and G , F_θ are distribution functions of g and f_θ respectively. Using the Basu–Lindsay approach, the restricted minimum divergence estimator is to be obtained by minimizing the S -divergence between g_n^* and f_θ^* , subject to the restriction (2). Thus, the corresponding estimating equation is given by

$$\left. \begin{aligned} \int K(\delta_n^*(x)) f_\theta^*(x)^{1+\alpha} \widetilde{u}_\theta(x) dx + H(\theta) \lambda_n &= 0 \\ h(\theta) &= 0 \end{aligned} \right\}. \quad (4)$$

where $\delta_n^*(x) = \frac{g_n^*(x)}{f_\theta^*(x)} - 1$ and $\widetilde{u}_\theta(x) = \nabla \log f_\theta^*(x)$. In general, the resulting estimator is not the same as the RMSDE obtained by minimizing $S_{(\alpha, \lambda)}(g_n^*, f_\theta)$ over Θ_0 ; we denote it as the restricted minimum S^* -divergence estimator (RMSDE*). We follow Ghosh and Basu (2014) to derive the asymptotic distribution of the RMSDE*. Let $\tilde{\theta}^{g^*} = \arg \min_{\theta \in \Theta_0} S_{(\alpha, \lambda)}(g^*, f_\theta)$ be the restricted “best fitting parameter” under g . Then we have the following theorem.

Theorem 2.2. *Consider the above set-up of continuous models and assume that the conditions (SB1)–(SB7) of Ghosh and Basu (2014), presented in the Supplementary material, hold with respect to $\Theta_0 = \{\theta : h(\theta) = 0\}$. Then,*

- (i) *There exists a consistent sequence $\tilde{\theta}_n^*$ of roots to the restricted minimum S^* -divergence estimating equations (4).*
- (ii) *Asymptotically, $\sqrt{n} (\tilde{\theta}_n^* - \tilde{\theta}^{g^*}) \sim N_p (0, \widetilde{P}_{\alpha, \lambda}^*(\tilde{\theta}^{g^*}) \widetilde{V}_{\alpha, \lambda}^*(g) \widetilde{P}_{\alpha, \lambda}^*(\tilde{\theta}^{g^*}))$, where $\widetilde{P}_{\alpha, \lambda}^*(\tilde{\theta}^{g^*})$ and $\widetilde{V}_{\alpha, \lambda}^*(g)$ are defined in Definition 1.2 in the Supplementary material.*

Now, for $g = f_{\theta_0}$ with $\theta_0 \in \Theta$ satisfying (2), we have $\tilde{\theta}^* = \theta_0$ and the asymptotic distribution of $\sqrt{n}(\tilde{\theta}_n^* - \theta_0)$ is normal with mean 0 and variance $\widetilde{P}_\alpha^*(\theta_0)\widetilde{V}_\alpha^*(\theta_0)\widetilde{P}_\alpha^*(\theta_0)$, where $\widetilde{P}_\alpha^*(\theta_0)$ and $\widetilde{V}_\alpha^*(\theta_0)$ are as in Definition 1.2 in the Supplementary material. Once again, the asymptotic distribution of the RMSDE* turns out to be independent of the parameter λ at the model.

Further, if we assume that the kernel used in smoothing is α -transparent for the restricted family $\mathcal{F}_0 = \{f_\theta : \theta \in \Theta_0\} \subset \mathcal{F}$ in the sense of Definition 9.1 of Ghosh and Basu (2014) presented in the Supplementary material (Definition 1.3), then it follows that for any $\theta_0 \in \Theta_0$, $\widetilde{J}_\alpha^*(\theta_0) = \widetilde{J}_\alpha(\theta_0)$, $\widetilde{V}_\alpha^*(\theta_0) = \widetilde{V}_\alpha(\theta_0)$ and so $\widetilde{P}_\alpha^*(\theta_0) = \widetilde{P}_\alpha(\theta_0)$. Hence at the model density $g = f_{\theta_0} \in \mathcal{F}_0$, the asymptotic distribution of the RMSDE* becomes exactly the same as that of the RMSDE under discrete model.

3. *S*-Divergence based Test (SDT) for Composite Hypothesis

Now we consider the problem of testing composite null hypothesis. Under the notations of the previous section, take a fixed (proper) subspace Θ_0 of the parameter space Θ . Our objective is to test for the hypothesis

$$H_0 : \theta \in \Theta_0 \quad \text{against} \quad H_1 : \theta \notin \Theta_0. \quad (5)$$

Following Ghosh et al. (2015), the *S*-divergence based test (SDT) statistics for testing the above hypothesis can be constructed as

$$\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_{\beta,\tau}, \widetilde{\theta}_{\beta,\tau}) = 2nS_{(\gamma,\lambda)}(f_{\widehat{\theta}_{\beta,\tau}}, f_{\widetilde{\theta}_{\beta,\tau}}), \quad (6)$$

where $S_{(\gamma,\lambda)}(\cdot, \cdot)$ is the *S*-divergence measure with parameter γ and λ and $\widetilde{\theta}_{\beta,\tau}$, $\widehat{\theta}_{\beta,\tau}$ denote the restricted (under Θ_0) and unrestricted MSDEs with tuning parameters β and τ . Their asymptotic distributions at the model are independent of τ . Indeed the estimators $\widehat{\theta}_{\beta,\tau_1}$ and $\widehat{\theta}_{\beta,\tau_2}$ are asymptotically equivalent in that

$$\sqrt{n}(\widehat{\theta}_{\beta,\tau_1} - \widehat{\theta}_{\beta,\tau_2}) \xrightarrow{\mathcal{P}} 0,$$

for any $\tau_1 \neq \tau_2$. We therefore replace $\widehat{\theta}_{\beta,\tau}$ and $\widetilde{\theta}_{\beta,\tau}$ by $\widehat{\theta}_{\beta,0}$ and $\widetilde{\theta}_{\beta,0}$ respectively, without altering the asymptotic properties of the test statistics in (6). The advantage of this substitution is that the latter set of estimators minimizes the DPD, and thus can be evaluated without any kernel smoothing. The asymptotic properties of the restricted MDPDE $\widehat{\theta}_\beta = \theta_{\beta,0}$ at the model $g = f_{\theta_0}$ is given by, see Basu et al. (2013b),

$$\sqrt{n}(\widetilde{\theta}_\beta - \theta_0) \xrightarrow{\mathcal{D}} N(0, \widetilde{P}_\beta(\theta_0)\widetilde{V}_\beta(\theta_0)\widetilde{P}_\beta(\theta_0)).$$

This distributional convergence holds under Conditions (D1)–(D5) of Basu et al. (2011) with respect to Θ_0 ; we refer to these 5 conditions as “Basu et al. conditions” throughout the rest of the paper. We also assume the standard conditions

of asymptotic inference, given by Assumptions A, B, C and D of Lehmann (1983, p. 429); we refer to them as the “Lehmann conditions”. Both set of conditions are presented in the supplementary material.

Now, to explore the asymptotic properties of the proposed (modified) SDT statistics $\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta) = \widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_{\beta,0}, \widetilde{\theta}_{\beta,0})$ for testing the composite hypothesis (5), we re-define the null parameter space Θ_0 in terms of r restrictions of the form (2). We also assume that the corresponding $p \times r$ matrix $H(\theta) = \frac{\partial h(\theta)}{\partial \theta}$ exists and it is a continuous function of θ with rank r . Indeed, this condition can be seen to hold for most parametric hypothesis. We start with the asymptotic null distribution of the proposed SDT.

Theorem 3.1. *Suppose the model density satisfies the Lehmann and Basu et al. conditions with respect to both Θ and Θ_0 and H_0 is true with $\theta_0 \in \Theta_0$ being the true parameter value. Then, the asymptotic null distribution of the SDT statistic $\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta)$ coincides with the distribution of $\sum_{i=1}^r \widetilde{\zeta}_i^{\gamma,\beta}(\theta_0) Z_i^2$, where Z_1, \dots, Z_r are independent standard normal variables, $\widetilde{\zeta}_1^{\gamma,\beta}(\theta_0), \dots, \widetilde{\zeta}_r^{\gamma,\beta}(\theta_0)$ are the nonzero eigenvalues of $A_\gamma(\theta_0) \widetilde{\Sigma}_\beta(\theta_0)$ with*

$$\widetilde{\Sigma}_\beta(\theta_0) = [J_\beta^{-1}(\theta_0) - P_\beta(\theta_0)] V_\beta(\theta_0) [J_\beta^{-1}(\theta_0) - P_\beta(\theta_0)]$$

$$\text{and } r = \text{rank} \left(V_\beta(\theta_0) [J_\beta^{-1}(\theta_0) - P_\beta(\theta_0)] A_\gamma(\theta_0) [J_\beta^{-1}(\theta_0) - P_\beta(\theta_0)] V_\beta(\theta_0) \right).$$

Noting the similarity of the above asymptotic null distribution with the case of testing simple null hypothesis, we can find critical values of the proposed SDT following Remark 3 of Basu et al. (2013a). We can also derive an asymptotic power approximation at any point $\theta^* \notin \Theta_0$; if $\theta^* \notin \Theta_0$ is the true parameter value then $\widehat{\theta}_\beta \xrightarrow{\mathcal{P}} \theta^*$ but $\widetilde{\theta}_\beta \xrightarrow{\mathcal{P}} \theta_0$ for some $\theta_0 \in \Theta_0$ with $\theta^* \neq \theta_0$. Define $\Sigma_\beta(\theta) = J_\beta^{-1}(\theta) V_\beta(\theta) J_\beta^{-1}(\theta)$. Then, by an argument similar to the one in the above theorem, one can show under the *Basu et al. conditions* that

$$\sqrt{n} \begin{pmatrix} \widehat{\theta}_\beta - \theta^* \\ \widetilde{\theta}_\beta - \theta_0 \end{pmatrix} \xrightarrow{\mathcal{D}} N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_\beta(\theta^*) & A_{12} \\ A_{12}^T & P_\beta(\theta_0) V_\beta(\theta_0) P_\beta(\theta_0) \end{bmatrix} \right),$$

for some $p \times p$ matrix $A_{12} = A_{12}(\theta^*, \theta_0)$. Further define

$$M_{1,\gamma,\lambda}(\theta^*, \theta_0) = \nabla S_{(\gamma,\lambda)}(f_\theta, f_{\theta_0})|_{\theta=\theta^*}, \quad M_{2,\gamma,\lambda}(\theta^*, \theta_0) = \nabla S_{(\gamma,\lambda)}(f_{\theta^*}, f_\theta)|_{\theta=\theta_0}.$$

Theorem 3.2. *Suppose the model density satisfies the Lehmann and Basu et al. conditions with respect to both Θ and Θ_0 and take any $\theta^* \notin \Theta_0$. An asymptotic approximation to the power function of the SDT statistic $\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta)$ for testing (5) at the significance level α is given by*

$$\widetilde{\pi}_{n,\alpha}^{\beta,\gamma,\lambda}(\theta^*) = 1 - \Phi \left(\frac{\sqrt{n}}{\widetilde{\sigma}_{\beta,\gamma,\lambda}(\theta^*, \theta_0)} \left(\frac{\widetilde{t}_{\alpha}^{\beta,\gamma}}{2n} - S_{(\gamma,\lambda)}(f_{\theta^*}, f_{\theta_0}) \right) \right), \quad \theta^* \neq \theta_0,$$

where $\widetilde{t}_{\alpha}^{\beta, \gamma}$ is the $(1 - \alpha)^{\text{th}}$ quantile of the asymptotic null distribution of the SDT $T_{\gamma, \lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta)$ and $\widetilde{\sigma}_{\beta, \gamma, \lambda}(\theta^*, \theta_0)^2 = M_{1, \gamma, \lambda}^T \Sigma_\beta M_{1, \gamma, \lambda} + M_{1, \gamma, \lambda}^T A_{12} M_{2, \gamma, \lambda} + M_{2, \gamma, \lambda}^T A_{12}^T M_{1, \gamma, \lambda} + M_{2, \gamma, \lambda}^T P_\beta V_\beta P_\beta M_{2, \gamma, \lambda}$.

The above theorem may be proved by a routine application of Taylor series and is omitted. This power approximation can help us to obtain the required sample size in any planned experiment to achieve a desired power. The theorem also shows that the proposed S -divergence based test is consistent for the composite hypotheses at any $\theta^* \notin \Theta_0$.

4. Robustness of the SDT for Composite Hypothesis

4.1. Influence Function of the Test

Let us define the statistical functional corresponding to the proposed SDT for the composite hypothesis as

$$\widetilde{T}_{\gamma, \lambda}^{(1)}(G) = S_{(\gamma, \lambda)}(f_{U_\beta(G)}, f_{\widetilde{U}_\beta(G)}),$$

where $U_\beta(G)$ is the MDPDE functional and $\widetilde{U}_\beta(G)$ is the restricted MDPDE functional under Θ_0 as defined in Ghosh (2014a). Consider the contaminated distribution $H_\epsilon = (1 - \epsilon)G + \epsilon \wedge_y$, where \wedge_y is the degenerate distribution at the contamination point y and ϵ is the contamination proportion. Then Hampel's first-order influence function (Hampel et al., 1986, Rousseeuw and Ronchetti, 1979, 1981) of the SDT functional $\widetilde{T}_{\gamma, \lambda}^{(1)}(G)$ is given by

$$\begin{aligned} IF(y; \widetilde{T}_{\gamma, \lambda}^{(1)}, G) &= \frac{\partial}{\partial \epsilon} \widetilde{T}_{\gamma, \lambda}^{(1)}(H_\epsilon) \Big|_{\epsilon=0} = M_{1, \gamma, \lambda}(U_\beta(G), \widetilde{U}_\beta(G))^T IF(y; U_\beta, G) \\ &\quad + M_{2, \gamma, \lambda}(U_\beta(G), \widetilde{U}_\beta(G))^T IF(y; \widetilde{U}_\beta, G), \end{aligned}$$

where $IF(y; U_\beta, G)$ and $IF(y; \widetilde{U}_\beta, G)$ are the influence functions (IFs) of U_β and $\widetilde{U}_\beta(G)$ respectively. Now, under the null hypothesis if $\theta_0 \in \Theta_0$ is the true value of parameter with $G = F_{\theta_0}$ then $U_\beta(F_{\theta_0}) = \theta_0$, $\widetilde{U}_\beta(F_{\theta_0}) = \theta_0$ and $M_{i, \gamma, \lambda}(\theta_0, \theta_0) = 0$ for all $i = 1, 2$; hence the first-order IF of our SDT statistic for the composite hypothesis also becomes zero at the null.

Therefore, to assess the robustness of the test, we consider the second order influence function of our statistic defined as $IF_2(y; \widetilde{T}_{\gamma, \lambda}^{(1)}, G) = \frac{\partial^2}{\partial^2 \epsilon} \widetilde{T}_{\gamma, \lambda}^{(1)}(H_\epsilon) \Big|_{\epsilon=0}$. In the particular case $G = F_{\theta_0}$ with $\theta_0 \in \Theta_0$, this second order influence function of the SDT simplifies to

$$IF_2(y; \widetilde{T}_{\gamma, \lambda}^{(1)}, F_{\theta_0}) = D_\beta(y; \theta_0)^T A_\gamma(\theta_0) D_\beta(y; \theta_0),$$

where $D_\beta(y; \theta_0) = [IF(y; U_\beta, F_{\theta_0}) - IF(y; \widetilde{U}_\beta, F_{\theta_0})]$. Note that the IF of the SDT at the composite null is also independent of λ and it is bounded if and only

if the influence function of the corresponding unrestricted and restricted MDPD functionals are both bounded or both diverge at the same rate. However, for most parametric models, these IFs of MDPDE and RMDPDE are seen to be bounded whenever $\beta > 0$ but unbounded at $\beta = 0$.

4.2. Level and Power Influence Functions

Now we consider the influence function of level and power of the SDT for composite hypothesis. Since the SDT is consistent, its asymptotic power is one at any fixed alternative; so we consider the asymptotic power under contiguous alternatives $H_{1,n} : \theta_n = \theta_0 + \frac{\Delta}{\sqrt{n}} \in \Theta - \Theta_0$ with $\Delta \in \mathbb{R}^p - \{0\}$ and $\theta_0 \in \Theta_0$. Clearly, to ensure the existence of such a θ_0 in Θ_0 there must exist a limit point θ_0 of the null parameter space Θ_0 ; we assume Θ_0 to be a closed subset of Θ . Next, following Hampel et al. (1986), we also consider the contaminations over these contiguous alternatives such that their effect tends to zero as θ_n tends to θ_0 at the same rate to avoid confusion between the null and alternative neighborhoods (also see Huber-Carol, 1970, Heritier and Ronchetti, 1994, Toma and Broniatowski, 2011). So, consider the contaminated distributions $F_{n,\epsilon,y}^L$ and $F_{n,\epsilon,y}^L$, defined in Definition 1.4 of the supplementary material, for level and power respectively and the level influence function (*LIF*) and the power influence function (*PIF*) as defined therein; also see Ghosh et al. (2015).

Let us first derive a general expression for asymptotic power $\tilde{P}(\Delta, \epsilon) = \lim_{n \rightarrow \infty} P_{F_{n,\epsilon,y}^P} \left(\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta) > \widetilde{t}_{\alpha}^{\beta,\gamma} \right)$ for testing composite hypothesis under contamination in the following theorem. Here, χ_p^2 denote a central chi-square random variable with p degrees of freedom and $\chi_{p,\delta}^2$ denote a non-central chi-square random variable with degrees of freedom p and non-centrality parameter δ .

Theorem 4.1. *Assume that the Lehmann and Basu et al. conditions hold for the model density and the null parameter space Θ_0 is such that there exists a limit point $\theta_0 \in \Theta_0$ satisfying $\theta_n = \theta_0 + \frac{\Delta}{\sqrt{n}} \in \Theta - \Theta_0$ for all $\Delta \in \mathbb{R}^p - \{0\}$. Then for any $\Delta \in \mathbb{R}^p$ and $\epsilon \geq 0$, we have the following:*

- (i) *The asymptotic distribution of $\widetilde{T}_{\gamma,\lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta)$ under $F_{n,\epsilon,y}^P$ is the same as that of the quadratic form $W^T A_\gamma(\theta_0) W$, where $W \sim N_p \left(\widetilde{\Delta}^*, \widetilde{\Sigma}_\beta(\theta_0) \right)$, where $\widetilde{\Delta}^* = \left[\Delta + \epsilon \left\{ IF(y; U_\beta, F_{\theta_0}) - IF(y; \widetilde{U}_\beta, F_{\theta_0}) \right\} \right]$. Equivalently, this distribution is the same as that of $\sum_{i=1}^r \widetilde{\zeta}_i^{\gamma,\beta}(\theta_0) \chi_{1,\widetilde{\delta}_i}^2$, where $\left(\sqrt{\widetilde{\delta}_1}, \dots, \sqrt{\widetilde{\delta}_p} \right)^T = \widetilde{V}_{\beta,\gamma}(\theta_0) \widetilde{\Sigma}_\beta^{-1/2}(\theta_0) \widetilde{\Delta}^*$ with $\widetilde{V}_{\beta,\gamma}(\theta_0)$ being the matrix of normalized eigenvectors of $A_\gamma(\theta_0) \widetilde{\Sigma}_\beta(\theta_0)$.*
- (ii) *$\widetilde{P}(\Delta, \epsilon) = \sum_{v=0}^{\infty} \widetilde{C}_v^{\gamma,\beta}(\theta_0, \widetilde{\Delta}^*) P \left(\chi_{r+2v}^2 > \frac{\widetilde{t}_{\alpha}^{\beta,\gamma}}{\widetilde{\zeta}_{(1)}^{\gamma,\beta}(\theta_0)} \right)$, where $\widetilde{\zeta}_{(1)}^{\gamma,\beta}(\theta_0)$ is the minimum of $\widetilde{\zeta}_i^{\gamma,\beta}(\theta_0)$ s over $i = 1, \dots, r$ and $\widetilde{C}_v^{\gamma,\beta}(\theta_0, \widetilde{\Delta})$ is as Definition 1.5 in the Supplementary material.*

Corollary 4.2. Putting $\epsilon = 0$ in above theorem, we get the asymptotic power under the contiguous alternatives $H_{1,n} : \theta = \theta_n = \theta_0 + \frac{\Delta}{\sqrt{n}}$ as

$$\widetilde{P}_0 = \widetilde{P}(\Delta, \epsilon = 0) = \sum_{v=0}^{\infty} \widetilde{C}_v^{\gamma, \beta}(\theta_0, \Delta) P \left(\chi_{r+2v}^2 > \widetilde{t}_{\alpha}^{\beta, \gamma} / \widetilde{\zeta}_{(1)}^{\gamma, \beta}(\theta_0) \right).$$

Corollary 4.3. Putting $\Delta = 0$ in above theorem, we get the asymptotic level under the probability distribution $F_{n,\epsilon,y}^L$ as

$$\widetilde{\alpha}_{\epsilon} = \widetilde{P}(\Delta = 0, \epsilon) = \sum_{v=0}^{\infty} \widetilde{C}_v^{\gamma, \beta}(\theta_0, \epsilon D_{\beta}(y, \theta_0)) P \left(\chi_{r+2v}^2 > \widetilde{t}_{\alpha}^{\beta, \gamma} / \widetilde{\zeta}_{(1)}^{\gamma, \beta}(\theta_0) \right).$$

Further, if we also take $\epsilon = 0$, then $F_{n,\epsilon,y}^L$ coincides with the null distribution and the asymptotic distribution of the proposed SDT obtained from part (i) of the above theorem coincides with asymptotic null distribution obtained independently in Theorem 3.1; hence $\widetilde{\alpha}_0 = \alpha$, as expected.

In practice, we can use finite truncation to approximate the infinite series in the above theorem, as discussed in Remark 3.1 of Ghosh et al. (2015).

Finally we will compute the level and power influence function of the proposed S -divergence based test statistics for composite hypothesis from the expression of $\widetilde{P}(\Delta, \epsilon)$ as obtained in Theorem 3.2. In particular, the power influence function (PIF) comes from a simple differentiation of $\widetilde{P}(\Delta, \epsilon)$ at $\epsilon = 0$ and then the level influence function (LIF) can be derived just by substituting $\Delta = 0$. The following theorem presents the form of the PIF and LIF of the proposed SDT. Clearly, both the LIF and PIF can be seen to be bounded whenever the influence function of the MDPDE under the null and overall parameter space both are bounded or both diverges at the same rate; this in turn implies the size and power robustness of the proposed SDT for $\beta > 0$.

Theorem 4.4. Assume that the Lehmann and Basu et al. conditions hold for the model density and the influence function $IF(y; U_{\beta}, F_{\theta_0})$ of the minimum DPD estimator is bounded. Then the power and level influence function of the proposed test statistics for composite hypothesis have the forms

$$PIF(y; T_{\gamma, \lambda}^{(1)}, F_{\theta_0}) = \frac{\partial}{\partial \epsilon} \widetilde{P}(\Delta, \epsilon) \Big|_{\epsilon=0} = D_{\beta}(y, \theta_0)^T C^*(\Delta, \theta_0, \gamma, \beta, \alpha)$$

and $LIF(y; T_{\gamma, \lambda}^{(1)}, F_{\theta_0}) = D_{\beta}(y, \theta_0)^T C^*(0, \theta_0, \gamma, \beta, \alpha),$

where $C^*(\Delta, \theta_0, \gamma, \beta, \alpha) = \sum_{v=0}^{\infty} \left[\frac{\partial}{\partial t} \widetilde{C}_v^{\gamma, \beta}(\theta_0, t) \Big|_{t=\Delta} \right] P \left(\chi_{r+2v}^2 > \frac{\widetilde{t}_{\alpha}^{\beta, \gamma}}{\widetilde{\zeta}_{(1)}^{\gamma, \beta}(\theta_0)} \right).$

5. Example: Testing Normal Mean with unknown variance

Now, we illustrate the proposed theory in the context of testing the mean of a normal distribution with unknown variance; the same problem with known variance is illustrated in Ghosh et al. (2015). Suppose we have a sample X_1, \dots, X_n

of size n from a population having a univariate normal density $N(\mu, \sigma^2)$ with both the parameters unknown. Based on this sample, we want to test the hypothesis $H_0 : \mu = \mu_0$ for a pre-specified real number μ_0 ; σ is not assumed to be known. Suppose $\widehat{\theta}_\beta = (\widehat{\mu}_\beta, \widehat{\sigma}_\beta)$ is the MDPDE of $\theta = (\mu, \sigma)$ with tuning parameter β and the corresponding restricted MDPDE under the above null is $\widetilde{\theta}_\beta = (\mu_0, \widetilde{\sigma}_\beta)$. Then a simple calculation shows that the proposed SDT for testing above H_0 is given by

$$\widetilde{T}_{\gamma, \lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta) = \frac{2n\widetilde{\kappa}_\gamma}{AB} \left[\frac{A}{\widetilde{\sigma}_\beta^\gamma} + \frac{B}{\widetilde{\sigma}_\beta^\gamma} - \frac{(1+\gamma)^{\frac{3}{2}} \widetilde{\sigma}_\beta^{1-B} \widetilde{\sigma}_\beta^{1-A}}{\sqrt{B\widetilde{\sigma}_\beta^2 + A\widetilde{\sigma}_\beta^2}} e^{-\frac{AB(\widehat{\mu}_\beta - \mu_0)^2}{2(B\widetilde{\sigma}_\beta^2 + A\widetilde{\sigma}_\beta^2)}} \right],$$

for $A, B \neq 0$, where $\widetilde{\kappa}_\gamma = (2\pi)^{-\frac{\gamma}{2}}(1+\gamma)^{-\frac{1}{2}}$. However, at $A = 0$ or $B = 0$, the above test statistic is defined in a limiting sense as

$$\begin{aligned} \widetilde{T}_{\gamma, \lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta) &= \frac{n\widetilde{\kappa}_\gamma}{\widetilde{\sigma}_\beta^\gamma} \left[\log \left(\frac{\widetilde{\sigma}_\beta^2}{\widetilde{\sigma}_\beta^2} \right) + \frac{1}{1+\gamma} \left(\frac{\widetilde{\sigma}_\beta^2}{\widetilde{\sigma}_\beta^2} - 1 \right) + \frac{(\widehat{\mu}_\beta - \mu_0)^2}{\widetilde{\sigma}_\beta^2} \right], \quad A = 0, \\ &= \frac{n\widetilde{\kappa}_\gamma}{\widetilde{\sigma}_\beta^\gamma} \left[\log \left(\frac{\widetilde{\sigma}_\beta^2}{\widetilde{\sigma}_\beta^2} \right) + \frac{1}{1+\gamma} \left(\frac{\widetilde{\sigma}_\beta^2}{\widetilde{\sigma}_\beta^2} - 1 \right) + \frac{(\widehat{\mu}_\beta - \mu_0)^2}{\widetilde{\sigma}_\beta^2} \right], \quad B = 0. \end{aligned}$$

In particular, $\gamma = \lambda = \beta = 0$, $\widetilde{\kappa}_0 = 1$, $\widehat{\theta}_0 = (\bar{X}, \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2)$, the unrestricted MLE of θ and $\widetilde{\theta}_0 = (\mu_0, \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2)$, the restricted MLE of θ under the null hypothesis. Then, the SDT statistic further simplifies to

$$\widetilde{T}_{0,0}^{(1)}(\widehat{\theta}_0, \theta_0) = n \log \left(\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2} \right);$$

this is again the likelihood ratio test statistic for the problem under consideration. Thus, the proposed SDT is a robust generalization of the LRT.

From Theorem 3.1, the asymptotic null distribution of the SDT statistics $\widetilde{T}_{\gamma, \lambda}^{(1)}(\widehat{\theta}_\beta, \widetilde{\theta}_\beta)$ for the composite null is also given by the distribution of $\zeta_1^{\gamma, \beta} Z_1$, where $Z_1 \sim \chi_1^2$ and $\zeta_1^{\gamma, \beta} = \frac{\kappa_\gamma v_\beta}{\sigma^{\gamma+2}}$ with $v_\beta = \frac{(1+\beta)^3}{(1+2\beta)^{3/2}} \sigma^2$ being the asymptotic variance of the MDPDE $\widehat{\mu}_\beta$. Note that, this asymptotic null distribution is exactly the same as that in case of testing simple hypothesis with known σ . In fact, all the asymptotic properties and the influence function of the test statistics for this case of composite hypothesis turns out to be exactly the same as obtained in the case of known σ (Ghosh et al., 2015); the main reason behind this is the asymptotic independence of the MDPDE or RMDPDE of μ and σ under the normal model.

5.1. A Real data Example: Telephone-fault data

Consider an interesting real dataset containing the records on telephone line faults presented and analyzed by Welch (1987); also studied by Simpson (1989) and Basu et al. (2013a,b). Table 1 presents the data on the ordered differences

between the inverse rates of test and control in 14 matched pairs of areas. These data could have been modeled by the normal distribution with mean μ and standard deviation σ if not for the first observation which produces a huge outlier with respect to the remaining 13 observations. The presence of this outlying observation changes the MLE of the parameters μ and σ drastically whereas the MDPDE with a slightly larger tuning parameter β produces robust estimators (Basu et al., 2013a,b); Table 2 presents the MDPDE and MLE (it is the MDPDE with $\beta = 0$) of the parameters under the full data and also after removing the outlying first observation.

Table 1: Telephone-fault data

Pair	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Difference	-988	-135	-78	3	59	83	93	110	189	197	204	229	289	310

Table 2: MDPDEs of μ and σ for the Telephone-fault data

	β	0	0.05	0.1	0.2	0.5
Full Data	$\hat{\mu}$	40.357	62.804	115.435	125.861	143.085
	$\hat{\sigma}$	311.332	273.909	148.766	120.105	96.564
Outlier	$\hat{\mu}$	119.462	120.844	122.361	125.893	143.085
Deleted data	$\hat{\sigma}$	129.532	127.406	125.128	120.009	96.564

For the present data set we consider the problem of testing two different hypothesis on the mean parameter μ , namely $H_0 : \mu = 0$ and $H'_0 : \mu = 115$ against their respective omnibus alternative. We consider the two cases of known and unknown σ ; for the known σ case we use its robust estimator 132 for the value of σ as suggested by Basu et al. (2013a,b). However, due to the non-robust nature of the MLE, the likelihood ratio test (and equivalently the traditional z -test or t -test) fails to reject the null H_0 but rejects the null H'_0 due to the presence of the large outlier; on the other hand after the removal of the large outlier the same test fails to reject the null H'_0 but soundly rejects H_0 . Here we apply the proposed S -divergence based test for these two testing problems using the formulas given in Ghosh et al. (2015) and in the present paper for known and unknown σ respectively. The p-values obtained by the S -divergence based tests (SDT) for the full data are presented in Figure 1 along with that corresponding to LRT. The robust nature of the proposed SDT is clear for both the cases.

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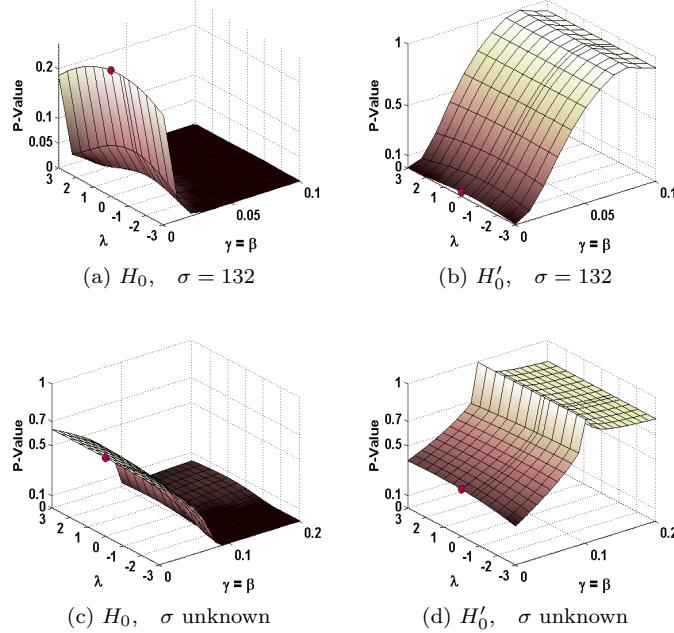


Figure 1: P-values for the SDT on the Telephone-fault data. The red point denotes the p-value for corresponding LRT

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